King Saud University
College of Science
Department of Mathematics

M 106 - INTEGRAL CALCULUS

Solutions of the first midterm exam
First Semester 1432-1433 H

Dr. Tariq A. AlFadhel[1][2]

[1]E-mail : alfadhel@ksu.edu.sa
[2]URL : http://faculty.ksu.edu.sa/alfadhel

Multiple choice questions (One mark for each question)

Question 1. The sum $\sum_{k=1}^{n}(3+k)^2$ is equal to :

(a). $\frac{1}{6}\left(2n^3+21n^2+54n\right)$

(b). $\frac{1}{6}\left(n^3+21n^2+73n\right)$

(c). $\frac{1}{6}\left(2n^3+19n^2+73n\right)$

(d). $\frac{1}{6}\left(2n^3+21n^2+73n\right)$

Answer: $\sum_{k=1}^{n}(3+k^2)=\sum_{k=1}^{n}(9+6k+k^2)=\sum_{k=1}^{n}9+6\sum_{k=1}^{n}k+\sum_{k=1}^{n}k^2$
$=9n+6\ \frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}=9n+3n^2+3n+\frac{2n^3+n^2+2n^2+n}{6}$
$=3n^2+12n+\frac{2n^3+3n^2+n}{6}=\frac{18n^2+72n+2n^3+3n^2+n}{6}$
$=\frac{2n^3+21n^2+73n}{6}=\frac{1}{6}\left(2n^3+21n^2+73n\right)$
The right answer is (d)

Question 2. The value of the integral $\int \sin(1+3x)\,dx$ is equal to :

(a). $-\frac{1}{3}\cos(1+3x)+c$

(b). $3\cos(1+3x)+c$

(c). $\frac{1}{3}\cos(1+3x)+c$

(d). $-\cos(1+3x)+c$

Answer: $\int \sin(1+3x)\,dx=\frac{1}{3}\int \sin(1+3x)\ 3\,dx=-\frac{1}{3}\cos(1+3x)+c$
The right answer is (a)

Question 3. The number z that satisfies the conclusion of the Mean Value Theorem for $f(x)=x^2$ on $[-2,0]$ is :

(a). $-\sqrt{\frac{8}{3}}$

(b). $\sqrt{\frac{8}{3}}$

(c). $\frac{-2}{\sqrt{3}}$

(d). $\dfrac{2}{\sqrt{3}}$

Answer : $f(z) = \dfrac{\int_{-2}^{0} x^2\ dx}{0-(-2)}$

$z^2 = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{0} = \frac{1}{2}\left[0 - \left(\frac{-8}{3}\right)\right] = \frac{1}{2}\ \frac{8}{3} = \frac{4}{3}$

$z = \sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

Note that $-\frac{2}{\sqrt{3}} \in [-2, 0]$ but $\frac{2}{\sqrt{3}} \notin [-2, 0]$

The right answer (c)

Question 4. The average value of $f(x) = \sqrt{x+1}$ on $[-1, 0]$ is equal to :

(a). $\dfrac{-3}{2}$

(b). $\dfrac{-2}{3}$

(c). $\dfrac{2}{3}$

(d). $\dfrac{3}{2}$

Answer : $f_{av} = \dfrac{\int_{-1}^{0} \sqrt{x+1}\ dx}{0-(-1)} = \int_{-1}^{0} (x+1)^{\frac{1}{2}}\ dx$

$f_{av} = \frac{2}{3}\left[(x+1)^{\frac{3}{2}}\right]_{-1}^{0} = \frac{2}{3}\left[(0+1)^{\frac{3}{2}} - (-1+1)^{\frac{3}{2}}\right] = \frac{2}{3}\ (1-0) = \frac{2}{3}$

The right answer is (c)

Question 5. If $F(x) = \int_{x}^{2x} f'(t)\ dt$ then $F'(x)$ is equal to :

(a). $f(2x) - f(x)$

(b). $2f(2x) - f(x)$

(c). $2f'(x)$

(d). $2f'(x) - f'(x)$

Answer : $F'(x) = \frac{d}{dx}\int_{x}^{2x} f'(t)\ dt = f'(2x)\ (2) - f'(x)\ (1) = 2f'(2x) - f'(x)$

The right answer is (d)

Question 6. The value of the integral $\int \frac{5^{\cosh x}}{cschx}\ dx$ is equal to :

(a). $5^{\cosh x} + c$

(b). $(\ln 5)\ 5^{\sinh x} + c$

(c). $\dfrac{5^{\cosh x}}{\ln 5} + c$

(d). $\dfrac{5^{\sinh x}}{\ln 5} + c$

Answer : $\displaystyle\int \frac{5^{\cosh x}}{cschx}\, dx = \int 5^{\cosh x} \sinh x \, dx = \frac{5^{\cosh x}}{\ln 5} + c$
The right answer is (c)

Question 7. The derivative of the function $f(x) = \cosh^{-1}(\sqrt{x})$ is equal to :

(a). $\dfrac{1}{2\sqrt{x^2 - x}}$

(b). $\dfrac{1}{\sqrt{2x^2 - x}}$

(c). $\dfrac{1}{2x\sqrt{x + 1}}$

(d). $\dfrac{1}{2x\sqrt{x^2 - 1}}$

Answer : $f'(x) = \dfrac{1}{\sqrt{(\sqrt{x})^2 - 1}} \ \dfrac{1}{2\sqrt{x}}$
$= \dfrac{1}{2\sqrt{x}\sqrt{x-1}} = \dfrac{1}{2\sqrt{x(x-1)}} = \dfrac{1}{2\sqrt{x^2 - x}}$
The right answer is (a)

Question 8. The value of the integral $\displaystyle\int (\sin x)(\sec x)^2 \, dx$ is equal to :

(a). $\dfrac{1}{\cos x} + c$

(b). $\dfrac{1}{\sin x} + c$

(c). $\dfrac{1}{\sec x} + c$

(d). $\dfrac{1}{3}(\sec x)^3 + c$

Answer : $\displaystyle\int (\sin x)(\sec x)^2 \, dx = \int \frac{\sin x}{(\cos x)^2}\, dx = -\int (\cos x)^{-2}(-\sin x)\, dx$
$= -\dfrac{(\cos x)^{-1}}{-1} + c = \dfrac{1}{\cos x} + c$
The right answer is (a)

Question 9. If $\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\, dx = f(x) + c$, then $f(x)$ is equal to :

(a). $e^{\cos^{-1} x}$

(b). $e^{-\cos^{-1} x}$

(c). $-e^{\cos^{-1} x}$

(d). $e^{\sin^{-1} x}$

Answer : $\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\, dx = -\int e^{\cos^{-1} x} \frac{-1}{\sqrt{1-x^2}}\, dx = -e^{\cos^{-1} x} + c$
The right answer is (c).

Question 10. The value of the integral $\int \frac{e^{2x}}{e^{4x}-1}\, dx$ is equal to :

(a). $\frac{1}{2} \sin^{-1}\left(e^{2x}\right) + c$

(b). $\frac{1}{2} \sinh^{-1}\left(e^{2x}\right) + c$

(c). $\frac{1}{2} \cosh^{-1}\left(e^{2x}\right) + c$

(d). $\cosh^{-1}\left(e^{2x}\right) + c$

Answer : $\int \frac{e^{2x}}{e^{4x}-1}\, dx = \frac{1}{2} \int \frac{2e^{2x}}{\sqrt{(e^{2x})^2 - (1)^2}}\, dx = \frac{1}{2} \cosh^{-1}(e^{2x}) + c$
The right answer is (c)

Full questions

Question 11. Approximate the integral $\int_0^1 e^{4x} dx$ using **Simpson's rule** for $n = 4$. [3 marks]

Answer :
$f(x) = e^{4x}$, $[a, b] = [0, 1]$ and $n = 4$.
$\Delta x = \frac{1-0}{4} = 0.25$
$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$ and $x_4 = 1$

$$\int_0^1 e^{4x} dx \approx \frac{1-0}{3(4)} \left[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right]$$

$$= \frac{1}{12} \left[1 + 4(2.7183) + 2(7.3891) + 4(20.086) + 54.598\right]$$

$$= \frac{1}{12} \left[1 + 10.873 + 14.778 + 80.344 + 54.598\right] = \frac{1}{12}[161.59] = 13.466$$

$$\int_0^1 e^{4x} dx \approx 13.466$$

Question 12. If $y = (\cosh x)^{2x+1}$, then find y'. [2 marks]
Answer : $y = (\cosh x)^{2x+1} \Rightarrow \ln y = \ln (\cosh x)^{2x+1} = (2x+1)\ln(\cosh x)$

Differentiate both sides

$\dfrac{y'}{y} = 2\ln(\cosh x) + (2x+1)\dfrac{\sinh x}{\cosh x} = 2\ln(\cosh x) + (2x+1)\tanh x$

$y' = y\left[2\ln(\cosh x) + (2x+1)\tanh x\right]$

$y' = (\cosh x)^{2x+1}\left[2\ln(\cosh x) + (2x+1)\tanh x\right]$

Question 13. Evaluate the integral $\displaystyle\int \frac{x-2}{\sqrt{8-2x^2}}\,dx$ [3 marks]

Answer :

$$\int \frac{x-2}{\sqrt{8-2x^2}}\,dx = \int \frac{x}{\sqrt{8-2x^2}}\,dx - \int \frac{2}{\sqrt{8-2x^2}}\,dx$$

$$= \int (8-2x^2)^{-\frac{1}{2}}\,x\,dx - 2\int \frac{1}{\sqrt{\left(\sqrt{8}\right)^2 - \left(\sqrt{2}x\right)^2}}\,dx$$

$$= -\frac{1}{4}\int (8-2x^2)^{-\frac{1}{2}}(-4x)\,dx - \frac{2}{\sqrt{2}}\int \frac{\sqrt{2}}{\sqrt{\left(2\sqrt{2}\right)^2 - \left(\sqrt{2}x\right)^2}}\,dx$$

$$= -\frac{1}{4}\frac{(8-2x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{2}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x}{2\sqrt{2}}\right) + c$$

$$= -\frac{1}{2}\sqrt{8-2x^2} - \frac{2}{\sqrt{2}}\sin^{-1}\left(\frac{x}{2}\right) + c$$

Question 14. Evaluate the integral $\displaystyle\int \frac{1}{x\sqrt{4+(\ln x)^2}}\,dx$ [2 marks]

Answer :

$$\int \frac{1}{x\sqrt{4+(\ln x)^2}}\,dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{(2)^2+(\ln x)^2}}\,dx = \sinh^{-1}\left(\frac{\ln x}{2}\right) + c$$
