M 106 - INTEGRAL CALCULUS

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Solution of the first mid-term exam

First semester 1433-1434 H

Multiple choice questions (One mark for each question)

Q.1 $\dfrac{d}{dx}\displaystyle\int_{-x}^{1} f(t)\ dt$ equals to :

(a) $f(x)$ (b) $-f(x)$ (c) $f(-x)$ (d) $-f(-x)$

Answer : Using Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{-x}^{1} f(t)\ dt = 0 - f(-x)\ (-1) = f(-x)$$

The right answer is (c)

Q.2 The value of the integral $\displaystyle\int_{1}^{4} \left(2+3\sqrt{x}\right)\ dx$ is equal to

(a) 19 (b) 20 (c) 21 (d) 22

Answer : $\displaystyle\int_{1}^{4} \left(2+3\sqrt{x}\right)\ dx = \int_{1}^{4}\left(2+3x^{\frac{1}{2}}\right)\ dx = \left[2x+3\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= \left[2x+2\left(\sqrt{x}\right)^3\right]_1^4 = \left[2(4)+2\left(\sqrt{4}\right)^3\right] - \left[2(1)+2\left(\sqrt{1}\right)^3\right]$

$= (8+16)-(2+2) = 20$

The right answer is (b)

Q.3 By using the integral $\displaystyle\int_{0}^{3}\left(4-\frac{x^2}{4}\right)\ dx = \frac{39}{4}$. Then the number z that satisfies the conclusion of the Mean Value Theorem for this integral will be

(a) $\sqrt{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{3}{2}$ (d) $\sqrt{\dfrac{3}{2}}$

Answer : Note that $f(x) = 4 - \dfrac{x^2}{4}$ and $[a,b] = [0,3]$

According to the Mean Value Theorem there exists $z \in (0,3)$ such that

$$f(z) = \frac{\displaystyle\int_{0}^{3}\left(4-\frac{x^2}{4}\right)\ dx}{3-0}$$

$$4-\frac{z^2}{4} = \frac{\frac{39}{4}}{3} = \frac{39}{12}$$

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$\frac{z^2}{4} = 4 - \frac{39}{12} = \frac{48-39}{12} = \frac{9}{12} = \frac{3}{4}$

$z^2 = 3 \Rightarrow z = \pm\sqrt{3}$ (Note that $-\sqrt{3} \notin (0,3)$)

Hence $z = \sqrt{3}$

The right answer is (a)

Q.4 The sum $\sum_{k=1}^{n}\left(3k^2 - \frac{1}{2}\right)$ is equal to

(a) $\frac{1}{2}n^2(2n+3)$ (b) $\frac{1}{2}n^2(2n+2)$
(c) $\frac{1}{2}n^2(2n+1)$ (d) $\frac{1}{3}n^2(2n+1)$

The answer : $\sum_{k=1}^{n}\left(3k^2 - \frac{1}{2}\right) = 3\sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}\frac{1}{2}$

$= 3\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}n = \frac{n(n+1)(2n+1)}{2} - \frac{n}{2}$

$= \frac{n}{2}[(n+1)(2n+1) - 1] = \frac{n}{2}\left[\left(2n^2+3n+1\right) - 1\right] = \frac{n}{2}(2n^2+3n)$

$= \frac{n^2}{2}(2n+3) = \frac{1}{2}n^2(2n+3)$

The right answer is (a)

Q.5 Find x such that $x = \log(1) - \log(10) + 1$

(a) ≈ -1.3026 (b) 0 (c) 1 (d) 2

The answer : Note that $\log(1) = 0$ and $\log(10) = 1$

$x = \log(1) - \log(10) + 1 = 0 - 1 + 1 = 0$

The right answer is (b)

Q.6 The derivative of the function $f(x) = \tanh^{-1}(x^2-1)$ is equal to

(a) $\frac{x}{x^2-x^4}$ (b) $\frac{2x}{2x^2-x^4}$ (c) $\frac{2x}{x^2-2x^4}$ (d) $\frac{2x}{x^2-4x^4}$

The answer : $f(x) = \tanh^{-1}(x^2-1) = \frac{2x}{1-(x^2-1)^2}$

$= \frac{2x}{1-(x^4-2x^2+1)} = \frac{2x}{1-x^4+2x^2-1} = \frac{2x}{2x^2-x^4}$

The right answer is (b)

Q.7 Find $\int \sec x(\sec x + \tan x)\, dx$:

(a) $\tan x + \sec x + c$ (b) $\tan x + \csc x + c$
(c) $\cot x + \csc x + c$ (d) $\cot x + \sec x + c$

The answer : $\int \sec x(\sec x + \tan x)\ dx = \int \left(\sec^2 x + \sec x\ \ \tan x\right)\ dx$

$= \int \sec^2 x\ dx + \int \sec x \tan x\ dx = \tan x + \sec x + c$

The right answer is (a)

Q.8 Assume that $\dfrac{dy(x)}{dx} = \sin(x+2)$, then

(a) $y(x) = \dfrac{1}{2}\cos(x+2) + c$ (b) $y(x) = -\dfrac{1}{2}\cos(x+2) + c$

(c) $y(x) = -\cos(x+2) + c$ (d). $y(x) = \cos(x+2) + c$

The answer : $\dfrac{dy(x)}{dx} = \sin(x+2) \Rightarrow y(x) = \int \sin(x+2)\ dx$

$y(x) = -\cos(x+2) + c$

The right answer is (c)

Q.9 The integral $\int \dfrac{1}{\sqrt{1-e^{2x}}}\ dx$ is equal to

(a) $-sech^{-1}\left(e^x\right) + c$ (b) $sech^{-1}\left(e^x\right) + c$

(c) $-\cosh^{-1}\left(e^x\right) + c$ (d) $\cosh^{-1}\left(e^x\right) + c$

The answer : $\int \dfrac{1}{\sqrt{1-e^{2x}}}\ dx = \int \dfrac{e^x}{e^x\sqrt{1-(e^x)^2}}\ dx$

$= -sech^{-1}\left(e^x\right) + c$

(Note : In this question $f(x) = e^x$ and $a = 1$) .

The right answer is (a)

Q.10 The integral $\int x2^{x^2}\ dx$ is equal to

(a) $\dfrac{1}{2}\dfrac{2^{x^2}}{\ln 2} + c$ (b) $\dfrac{2^{x^2}}{\ln 2} + c$ (c) $\dfrac{1}{2}\dfrac{2^{x^2}}{x\ln 2} + c$ (d) $\dfrac{2^{x^2}}{4} + c$

The answer : $\int x2^{x^2}\ dx = \dfrac{1}{2}\int 2^{x^2}\ 2x\ dx = \dfrac{1}{2}\dfrac{2^{x^2}}{\ln 2} + c$

The right answer is (a)

Full questions

Q.11 Approximate the integral $\int_1^2 \frac{1}{\sqrt{3+x^2}}\,dx$ using the **Trapezoidal rule** for a regular partition with $n = 4$. **[3 marks]**

The answer : $[a, b] = [1, 2]$, $n = 4$, and $f(x) = \frac{1}{\sqrt{3+x^2}}$.

$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$

$x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$.

$$\int_1^2 \frac{1}{\sqrt{3+x^2}}\,dx \approx \frac{2-1}{2(4)}\left[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)\right]$$

$$\approx \frac{1}{8}[0.5 + 2(0.468165)) + 2(0.436436) + 2(0.406138) + 0.377964]$$

$$\approx \frac{1}{8}[3.49944] \approx 0.43743$$

Q.12 If $y(x) = \left(x^4 + x^2 + 1\right)^{\ln(x^2+1)}$, then find $y'(x)$. **[4 marks]**

The answer :

$$y(x) = \left(x^4 + x^2 + 1\right)^{\ln(x^2+1)} \Rightarrow \ln|y(x)| = \ln(x^2+1)\ \ln(x^4 + x^2 + 1)$$

Differentiate both sides with respect to x

$$\frac{y'(x)}{y(x)} = \frac{2x}{x^2+1}\ \ln(x^4 + x^2 + 1) + \frac{4x^3 + 2x}{x^4 + x^2 + 1}\ \ln(x^2+1)$$

$$y'(x) = y(x)\left[\frac{2x}{x^2+1}\ \ln(x^4 + x^2 + 1) + \frac{4x^3 + 2x}{x^4 + x^2 + 1}\ \ln(x^2+1)\right]$$

$$y'(x) = \left(x^4 + x^2 + 1\right)^{\ln(x^2+1)}\left[\frac{2x\ln(x^4 + x^2 + 1)}{x^2+1} + \frac{(4x^3 + 2x)\ln(x^2+1)}{x^4 + x^2 + 1}\right]$$

Q.13 Evaluate the integral $\int \frac{1}{x\sqrt{x^8 - 25}}\,dx$. **[4 marks]**

The answer : $\int \frac{1}{x\sqrt{x^8-25}}\,dx = \int \frac{1}{x\sqrt{(x^4)^2 - (5)^2}}\,dx$

$$= \int \frac{x^3}{x^3\ x\sqrt{(x^4)^2 - (5)^2}}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{(x^4)^2 - (5)^2}}\,dx$$

$$= \frac{1}{4}\ \frac{1}{5}\ \sec^{-1}\left(\frac{x^4}{5}\right) + c = \frac{1}{20}\ \sec^{-1}\left(\frac{x^4}{5}\right) + c$$

(Note : In this question $f(x) = x^4$ and $a = 5$) .

Q.14 If $\int_{1}^{x} f(t)\ dt = x \ln|x|$, calculate $f\left(e^2\right)$. **[4 marks]**

The answer : Differentiate both sides with respect to x

$$\frac{d}{dx}\int_{-1}^{x} f(t)\ dt = \frac{d}{dx}\left(x \ln|x|\right)$$

$$f(x) = (1)\ \ln|x| + x\ \frac{1}{x} = 1 + \ln|x|$$

Note that $\frac{d}{dx}\int_{1}^{x} f(t)\ dt = f(x)$ by the Fundamental Theorem of calculus

$$f\left(e^2\right) = 1 + \ln\left(e^2\right) = 1 + 2\ln(e) = 1 + 2\ (1) = 1 + 2 = 3\ .$$
